A METHOD OF SOLVING SYSTEMS OF NONLINEAR EQUATIONS OF THE POTENTIAL TYPE

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S.V. SIMEONOV
(Sofiia, Bulgaria)

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Present paper considers systems of nonlinear equations in the case wnen they are a minimum condition for a functional, given in an n-dimensional Euclidean space E_n. A special method of descent is used to solve the system. This method entails a consecutive inclusion of unknowns, and the problem is solved, when the last unknown has been included. It is shown, that in the case of a linear system, the method of descent leads to a computational scheme of the method of bounding.

Investigation of a large number of physical problems, specially in mechanics and automation, leads to the necessity of solving systems of equations representing an extremal condition for some functional $f(x)$. These we shall call the systems of the potential type and we shall present a method of solution of such systems in the case when $f(x)$ is given in some region G of the n-dimensional Euclidean space E_n.

1. Suppose that a system of nonlinear equations

$$f_i(x) = 0 \quad (i = 1,\ldots,n) \qquad x = (x_1, x_2,\ldots,x_n) \in G \tag{1.1}$$

is given. We shall assume that the system (1.1) represents the condition for a minimum of the functional $f(x)$ which is bounded below and has a unique stationary point x^*. Obviously

$$f_i(x) = \partial f(x)/\partial x_i \tag{1.2}$$

In determining the point x^*, condition of potentiality of the operator $f_x = (f_1,\ldots,f_n)$ makes it possible to consider, irrespective of the method adopted in our investigation, the functional $f(x)$ instead of the system (1.1). This important property is widely utilised ([1 to 4] e.a.) in qualitative investigation of operators of this particular type defined on the arbitrarily general Banach spaces as well as in establishing effective methods of their solution.

Let $x^{(0)} \in G$ be a given initial point. Under our conditions $f(x^{(0)}) \geqslant f(x^*)$ where equality occurs only if $x^{(0)} = x^*$. We assume that the plan (strategy) of descent from $f(x^{(0)})$ which we are about to devise must be such, that $f(x^*)$ is reached quickly and with certainty. Similar methods ([4 and 5] e.a.) usually realise the descent along a straight line of a deepest slope issuing from $f(x^{(0)})$. Length of the path of descent in this direction is usually determined using various criteria, and these often lead to processes which converge fairly slowly, particularly during the approach of $f(x^*)$.

Here we have adopted another plan of descent which, in the case of two variables, we shall agree to call the descent along the "valleys (ravines)". These ravines are, however, different from those introduced by I.M. Gel'fand. Such a descent will, under our assumptions concerning the functional $f(x)$, always terminate at the required point of $f(x^*)$.

Let us realise, in the first stage, a descent from $f(x^{(0)})$ in the direction x_1 to the lowest point of $f(x^{(1)})$ where $x_i^{(1)} = x_i^{(0)}$ when $i \geqslant 2$. Obviously, $x_1^{(1)}$ will be a solution of Eq.

$$f_1(x_1; x_2^{(1)}, \ldots, x_n^{(1)}) = 0 \tag{1.3}$$

which we, at present, consider as an equation in one variable x_1.

We shall assume that point $x^{(1)}$ is situated at the bottom of a ravine. In order to effect a further descent along it, we must have another point $x^{(1)\prime}$. This point can also be obtained from (1.3) by adding to another coordinate, say x_2, an increment Δx_2. To do this, we must solve Eq.

$$f_1(x_1;\, x_2^{(0)} + \Delta x_2,\, x_3^{(0)}, \ldots, x_n^{(0)}) = 0 \tag{1.4}$$

Its root will be $x_1^{(1)\prime} = x_1^{(1)} + \Delta x_1$. Direction t of the further descent is given in terms of the parameter t, by Formulas

$$x_i(t) = x_i^{(1)} + \frac{\Delta x_i}{\Delta s}\, t \quad (i = 1, 2), \qquad x_i(t) = x_i^{(1)} = x_i^{(0)} \quad (i = 1, 2) \tag{1.5}$$
$$(\Delta s = \sqrt{(\Delta x_1)^2 + (\Delta x_2)^2})$$

We can also put $\Delta s = 1$.

Descent along t continues until the lowest point of $f(x(t^*))$ is reached. It is easily seen that t^* is a solution of

$$f_1(x(t))\, \Delta x_1 + f_2(x(t))\, \Delta x_2 = 0 \tag{1.6}$$

Left-hand part of this Eq. gives the increment of the functional along t, and the descent from $x^{(0)}$ to $x(t^*) = x^{(1,1)}$ shall be called the first subcycle.

Taking now $x^{(1,1)}$ as $x^{(0)}$ we complete the second subcycle in an analogous manner and repeat this process until $x^{(1,m)}$ and $x^{(1,m+1)}$ are sufficiently close to each other. We shall denote the limit point by $x^{(2)}$. Its existence follows from the previous assumptions concerning the functional.

We shall now show that the attained point $x^{(2)}$ is a solution of a set of two Eqs.

$$f_1(x_1, x_2;\, x_3^{(0)}, \ldots, x_n^{(0)}) = 0 \qquad f_2(x_1, x_2;\, x_3^{(0)}, \ldots, x_n^{(0)}) = 0 \tag{1.7}$$

Indeed, in order to be a solution of (1.7), $x^{(2)}$ must satisfy the condition $\Delta f = 0$, i.e. condition (1.6) when Δx_1 and Δx_2 are arbitrary and sufficiently small. This is the condition fulfilled by the process described above.

Since Δx_2 is a suitably chosen, sufficiently small magnitude, we can write (1.4) as

$$a_{11}\Delta x_1 + a_{12}\Delta x_2 = 0, \qquad \text{or} \qquad \Delta x_1 = -\frac{a_{12}}{a_{11}}\, \Delta x_2 \tag{1.8}$$

where

$$a_{11} = a_{11}(x^{(1)}) = \left.\frac{\partial f_1(x)}{\partial x_1}\right|_{x=x^{(1)}}, \qquad a_{12} = \left.\frac{\partial f_1(x)}{\partial x_2}\right|_{x=x^{(1)}} \tag{1.9}$$

provided that $f_1(x)$ admits partial derivatives in x_1 and x_2.

Coefficients (1.9) can also be determined by numerical differentiation and the latter method is particularly suitable when a computer is accessible.

Solution of equations of the type (1.6) shall be discussed below.

Let us now continue our process in the following manner. We shall add to the point $x^{(2)}$ a small increment Δx_3 and solve the system (1.7) once more

$$f_1(x_1, x_2;\, x_3^{(0)} + \Delta x_3,\, x_4^{(0)}, \ldots, x_n^{(0)}) = 0$$
$$f_2(x_1, x_2;\, x_3^{(0)} + \Delta x_3,\, x_4^{(0)}, \ldots, x_n^{(0)}) = 0 \tag{1.10}$$

This yields $x_1^{(2)\prime}$ and $x_2^{(2)\prime}$ which can be used to obtain $\Delta x_1 = x_1^{(2)\prime} - x_1^{(2)}$ and $\Delta x_2 = x_2^{(2)\prime} - x_2^{(2)}$. The obtained magnitudes together with the assumed Δx_3 will define new direction of descent t, which will have the following variable coordinates

$$x_i(t) = x_i^{(2)} + \frac{\Delta x_i}{\Delta s}\, t \quad (i = 1, 2, 3), \quad x_i(t) = x_i^{(2)} = x_i^{(0)} \quad (i > 3)$$
$$\Delta s = \sqrt{(\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2} \tag{1.11}$$

Value of the parameter t at which the functional $f(x)$ reaches its minimum on t, is given by the root of Eq.

$$f_1(x(t))\, \Delta x_1 + f_2(x(t))\, \Delta x_2 + f_3(x(t))\, \Delta x_3 = 0 \tag{1.12}$$

where $x(t)$ is given by (1.11). As before, this is the first subcycle which is repeated until a point $x^{(3)}$ is reached with a necessary accuracy. When some small value is assumed for Δx_3, we can use it to calculate Δx_1 and Δx_2 from a linear system

$$a_{11}\Delta x_1 + a_{12}\Delta x_2 = -a_{13}\Delta x_3$$
$$a_{21}\Delta x_1 + a_{22}\Delta x_2 = -a_{23}\Delta x_3 \tag{1.13}$$

where

$$a_{ik} = a_{ki} = \frac{\partial f_i(x)}{\partial x_k} = \frac{\partial f_k(x)}{\partial x_i}\bigg|_{x=x^{(2)}} \tag{1.14}$$

We assume here that functions f_i are differentiable.
Then, point $x^{(3)}$ is a solution of a set of three Eqs.

$$\begin{aligned} f_1(x_1, x_2, x_3; x_4^{(0)}, \ldots, x_n^{(0)}) &= 0 \\ f_2(x_1, x_2, x_3; x_4^{(0)}, \ldots, x_n^{(0)}) &= 0 \\ f_3(x_1, x_2, x_3; x_4^{(0)}, \ldots, x_n^{(0)}) &= 0 \end{aligned} \tag{1.15}$$

We shall call each inclusion of a new unknown, a cycle. Repeating the above process we reach the k-th cycle in which we solve the following nonlinear Eq.:

$$f_1(x(t))\Delta x_1 + f_2(x(t))\Delta x_2 + \ldots + f_k(x(t))\Delta x_k = 0 \tag{1.16}$$

where Δx_k is given as before, while $x(t)$ is defined in this subcycle by the Formulas

$$x_i(t) = x_i^{(k-1,\, s-1)} + \frac{\Delta x_i}{\Delta s}t \quad (i = 1, 2, \ldots, k) \tag{1.17}$$

$$x_i(t) = x_i^{(k-1,\, s-1)} = x_i^{(0)} \quad (i > k), \qquad \Delta s = \left(\sum_{j=1}^{k}(\Delta x_j)^2\right)^{1/2}$$

Here $\Delta x_i = x_i^{(k-1,\, s)} - x_i^{(k-1,\, s-1)} (i < k)$ where $x_i^{(k-1,\, s)}$ are solutions of

$$\begin{aligned} &f_1(x_1, x_2, \ldots, x_{k-1}; x_k^{(0)} + \Delta x_k, x_{k+1}^{(0)}, \ldots, x_m^{(0)}) = 0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &f_{k-1}(x_1, x_2, \ldots, x_{k-1}; x_k^{(0)} + \Delta x_k, x_{k+1}^{(0)}, \ldots, x_m^{(0)}) = 0 \end{aligned} \tag{1.18}$$

As before, Δx can be found from a linear system

$$\begin{aligned} &a_{11}\Delta x_1 + a_{12}\Delta x_2 + \ldots + a_{1,\, k-1}\Delta x_{k-1} = -a_{1k}\Delta x_k \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &a_{k-1,\, 1}\Delta x_1 + a_{k-1,\, 2}\Delta x_2 + \ldots + a_{k-1,\, k-1}\Delta x_{k-1} = -a_{k-1,\, k}\Delta x_k \end{aligned} \tag{1.19}$$

$$a_{ij} = a_{ji} = \frac{\partial f_i(x)}{\partial x_j} = \frac{\partial f_j(x)}{\partial x_i}\bigg|_{x=x^{(k-1,\, s-1)}} \tag{1.20}$$

If the computation is sufficiently accurate, then the n-th cycle gives the required solution. If, on the other hand, a computational error is committed so that $x^{(n)}$ does not satisfy (1.1) with sufficient degree of accuracy, or if perturbation of the functional $f(x)$ has occurred, we take $x^{(n)}$ as $x^{(0)}$ and repeat the whole process.

We can now see that the plan adopted by us, resulted in a method which we shall call the method of consecutive inclusion of unknowns.

It should be noted that our plan can also be realised in the case when the functional $f(x)$ is known, but not the system (1.1).

2. The proposed method requires multiple solution of nonlinear equations in one variable, of the form

$$y(t) \equiv f_1(x(t))\Delta x_1 + f_2(x(t))\Delta x_2 + \ldots + f_k(x(t))\Delta x_k = 0$$

$$x_i(t) = x_i' + a_i t \qquad (i = 1, 2, \ldots, k) \tag{2.1}$$

We shall now make an assumption, common in mechanics [2 and 3] that the functional $f(x)$ is convex in G and satisfies the condition $d^2 f(x) > \gamma^2 (dx)^2$. We shall show that under this condition $y(t)$ is always a monotonously increasing function. This will allow us to employ convenient methods of solution, always resulting in an exact solution of (2.1).

Let us write the differential of (2.1)

$$dy = \left[\sum_{i=1}^{k}\sum_{j=1}^{k} f_{ij}(x(t))\,\Delta x_i \Delta x_j\right]\frac{1}{\Delta s}\,dt \tag{2.2}$$

Conditions of convexity imply that the expression within the middle bracket will be a positive definite quadratic form for any $x \in G$. This gives us $dy > \gamma^2\, dt$ which was to be proved. Existence of a unique solution follows from the existence of $\gamma^2 > 0$.

We shall use a method of secants [6] combined with the method of regula falsi to solve

(2.1). We choose an initial approximation t_0, say $t_0 = 0$ and calculate $y_0 = y(t_0)$. Next we increase the variable t by an increment $\Delta t > 0$ and thus obtain $t_1 = t_0 + \Delta t$. Having found $y_1 = y(t_1)$, we obtain the remaining approximations by means of a recurrence formula

$$t_{i+1} = t_i - y_i \frac{t_i - t_{i-1}}{y_i - y_{i-1}} \tag{2.3}$$

When the function $y(t)$ has no point of inflection, then the above process is convergent and we easily see that all y_i are of the same sign. If this condition is not satisfied, then the process may diverge and in order to make it always converge, we must adopt the following procedure. Suppose that the sign of y_{k+1} differs from that of y_k. Then, denoting the exact solution by $t^* \in (t_k, t_{k+1})$ we continue our computation using the method of regula falsi. We write the Formula (2.3) in the form

$$t_{i+1} = t_i - y_i \frac{t_i - t_k}{y_i - y_k} \tag{2.4}$$

If changes of sign occur again, we replace t_k and y_k each time with those values, which immediately precede the change of sign.

When passing from t_i to t_{i+1}, we must make certain that the condition $x(t) \in G$ is fulfilled. If it is not, then we must replace t_{i+1} with t_{i+1}' such, that $t_i < t_{i+1}' < t_{i+1}$, $x(t_{i+1}') \in G$.

3. To illustrate the method we shall consider the case of a quadratic functional, when (1.1) is linear. In this case equations of the type (1.16) are also linear and can be solved by elementary methods. Subcycles are no longer necessary and this means that the bottom of the ravine runs along a straight line. Writing (1.1) as

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + b_1 = 0 \\ \cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + b_n = 0 \end{gathered} \tag{3.1}$$

and assuming that $x^{(0)} = 0$ we obtain, from the first equation,

$$x_1^{(1)} = \beta_{11}^{(1)} b_1 \qquad (\beta_{11}^{(1)} = -1/a_{11})$$

Further, by the linearity of our problem and putting $\Delta x_2 = 1$, we obtain from (1.8),

$$\Delta x_1^{(2)} = \beta_{11}^{(1)} a_{12}$$

Taking (1.8) into account we obtain, from (1.6) and (1.5)

$$x_2^{(2)} = \beta_{21}^{(2)} b_1 + \beta_{22}^{(2)} b_2, \quad x_1^{(2)} = x_1^{(1)} + \Delta x_1^{(1)} x_2^{(2)}$$

$$(\beta_{22}^{(2)} = -1/(a_{12}\Delta x_1^{(2)} + a_{22}),\ \beta_{21}^{(2)} = \Delta x_1^{(2)} \beta_{22}^{(2)})$$

Resulting $x_1^{(2)}$ and $x_2^{(2)}$ represent exact solutions of the system

$$a_{11}x_1 + a_{12}x_2 + b_1 = 0, \quad a_{21}x_1 + a_{22}x_2 + b_2 = 0$$

Let us now assume that we have obtained a solution of the $x^{(k-1)}$-th subsystem

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1,k-1}x_{k-1} + b_1 = 0 \\ \cdots\cdots\cdots\cdots\cdots \\ a_{k-1,1}x_1 + a_{k-1,2}x_2 + \ldots a_{k-1,k-1}x_{k-1} + b_{k-1} = 0 \end{gathered} \tag{3.2}$$

Next, putting $\Delta x_k^{(k)} = 1$, we obtain from (1.20) $\Delta x_i^{(k)}$

$$\begin{gathered} a_{11}\Delta x_1 + a_{12}\Delta x_2 + \ldots + a_{1,k-1}\Delta x_{k-1} + a_{1k} = 0 \\ \cdots\cdots\cdots\cdots\cdots \\ a_{k-1,1}\Delta x_1 + a_{k-1,2}\Delta x_2 + \ldots + a_{k-1,k-1}\Delta x_{k-1} + a_{k-1,k} = 0 \end{gathered} \tag{3.3}$$

We should note that both systems, (3.2) and (3.3) have an identical inverse matrix A_{k-1}^{-1}. From (1.17) and (3.3) we easily obtain

$$x_k^{(k)} = \sum_{j=1}^{k} \beta_{kj}^{(k)} b_j, \quad \beta_{kj}^{(k)} = -\frac{\Delta x_j^{(k)}}{\alpha^k}, \quad \alpha_k = \sum_{j=1}^{k} a_{jk}\Delta x_j \tag{3.4}$$

Clearly $\beta_{kj}^{(k)}$ are elements of the matrix A_k^{-1}. Remaining $\beta_{ij}^{(k)}$ are obtained from (1.17) by eliminating $t/\Delta s$

$$x_i^{(k)} = x_i^{(k-1)} + \Delta x_i^{(k)} x_k^{(k)} = \sum_{j=1}^{k} \beta_{ij}^{(k)} b_j, \quad \beta_{ij}^{(k)} = \beta_{ij}^{(k-1)} + \Delta x_i^{(k)} \beta_{kj}^{(k)}$$

$$(i, j = 1, 2, \ldots, k-1) \tag{3.5}$$

$\Delta x_i^{(k+1)}$ are computed by means of the same inverse matrix A_k^{-1}

$$\Delta x_i^{(k+1)} = \sum_{j=1}^{k} \beta_{ij}^{(k)} a_{j,k+1} \quad (i = 1, 2, \ldots, k), \quad \Delta x_{k+1}^{(k+1)} = 1 \tag{3.6}$$

Putting now $k = n$, we obtain the required solution x^* of (3.1).

This process of computation corresponds, as expected, to a wellknown method of bounding [7]. However, it was obtained in an entirely different manner, therefore the intermediate magnitudes have different meaning.

4. Various simplifying variants to the proposed method are possible. They include various processes of consecutive approximation in which the number of cycles increases together with the required accuracy. We shall consider two such variants.

We shall adopt the following plan of descent. First we perform the descent along x_1 and x_2 as in the Section 1. Then, taking the attained point $x^{(2)}$ as initial, we perform a similar descent along x_2 and x_3. Further, selecting various pairs of coordinates one after the other so as to include all coordinates in this process, we continue the descent until all the pairs give infinitesimal corrections. Coordinate pairs should be chosen so as to ensure the strongest possible inter dependence between the coordinates within each pair. Convergence of this process follows directly from the Theorem 2 of [2 and 3], or it can be proved using the method similar to that employed in Section 1. Computational scheme of this process is as follows.

Let us assume that descending along x_i and x_j we have reached a point $x^{(\nu)}$.

We select another pair containing one of the previously used coordinates, e.g. x_i, x_k. Then, choosing Δx_k in the manner used in Section 1, we obtain an expression analogous to (1.8)

$$\Delta x_i = -\frac{f_{ik}(x^{(\nu)})}{f_{ii}(x^{(\nu)})} \Delta x_k$$

and solve Eq.

$$f_i(x(t)) \Delta x_i + f_k(x(t)) \Delta x_k = 0$$

where

$$x_i(t) = x_i^{(\nu)} + \Delta x_i t, \quad x_k(t) = x_k^{(\nu)} + \Delta x_k t, \quad x_j(t) = x_j^{(\nu)} \quad (j \neq i, k)$$

This yields the point $x^{(\nu+1)}$. Selecting a new pair of coordinates we repeat the procedure e.t.c. This method obviates the necessity of solving repeatedly an increasing system of linear equations of the type (1.19), but may impair the rate of convergence.

We assume now that the unknowns can be split into several groups in such a manner, that each group contains those unknowns, which are most strongly interdependent. Then we arrange Eqs. of (1.1) into subsystems according to the same plan. Using the method of Section 1 we now solve, consecutively, each subsystem, using the final point of the subsystem just solved, as an initial point for the next subsystem.

The process terminates, when improvement in accuracy becomes vanishingly small.

Its convergence can be proved in a manner similar to the previous case.

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